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THERMODYNAMIC STUDY OF PLASMAS USING THE PRINCIPLE OF MAXIMUM ENTROPY

by Norbert Stankiewicz

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Cleveland, Ohio

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ABSTRACT

A maximum entropy (disorder) principle is used to determine the velocity distribution functions in a quiescent plasma (case 1), a current-carrying plasma (case 2), and a viscous plasma (case 3). The Saha equation (at the electron temperature) holds for case 1. and modified Saha equations hold for cases 2 and 3. The modification in case 2 is important only in situations where the directed energy of the electrons is a substantial fraction of the total energy. The results of case 3 are presented in a general form and need simplification before they can be applied to a specific situation.

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SUMMARY

The information theory idea of maximum entropy (disorder) is used to determine the most probable distribution functions of a species in a quiescent plasma (case 1), a current-carrying plasma (case 2), and a viscous plasma (case 3).

The Saha equation (at the electron temperature) holds for case 1; a modified Saha equation, which is dependent on the current density, holds for case 2; and a result, which is dependent on the off-diagonal elements of the pressure tensor, is obtained for case 3.

The modification in case 2 is important only in situations where the directed energy of the electrons is a substantial fraction of the total energy (i.e., electron diffusion Mach numbers, $\sim$1).

The results of case 3 are presented in a general form and need simplification before they can be applied to a specific situation. This case, however, is illustrative of the method presented herein and of its possible extension.

INTRODUCTION

The properties of a plasma can be expressed in terms of averages over the distribution function of its constituent species. Unfortunately these distribution functions are rarely either directly measurable or known beforehand. In some instances, the maximization of entropy offers a means by which these distribution functions can be deduced if certain constraints on the plasma are given. This report explores the application of the maximum entropy principle to various plasma states. In particular, the maximum entropy principle is applied to nonequilibrium systems.

Related studies include a paper by Holway (ref. 1), which contains a discussion of an elliptical probability function derivable from a maximum entropy principle. This function is used to construct a kinetic collision model; the Lagrange multipliers are identified in terms of the expansion coefficients of the distribution function (Chapman-Enskog expansion).

Kogan (ref. 2) investigates the relation between the most probable distribution function and the moments of the Boltzmann equation. The most probable distribution function is found by maximizing the entropy with the moments as constraints. His approach is similar to Holway's in that he compares the most probable distribution function with the Chapman-Enskog expanded distribution function.

A paper by Potapov (ref. 3) on the thermodynamics of a multispecie plasma with internal degrees of freedom is also of interest. He uses equilibrium thermodynamics to describe situations in which some species and degrees of freedom are not at the same temperature. His results, however, do not agree with the kinetic two-temperature results found in studies such as those of Bates, et al. (ref. 4).

In constrast to the previously cited work, this report is concerned with the application of the maximum entropy principle to systems having variables other than those found in the usual thermodynamic system.

According to information theory (see Jaynes, ref. 5), the most probable state of a statistical system is one in which the disorder is a maximum consistent with the given information about the system. (In the thermodynamic application of information theory, the equivalence of entropy and disorder is assumed.) Any measurement on the system or information about the system can only subtract from the total disorder. The information plays the role of a state function in the thermodynamic system and serves to constrain the system.

Information theory is equally applicable to equilibrium or nonequilibrium conditions. Equilibrium in the conventional sense becomes not a general result of the maximization of entropy, or disorder, but a result depending on specific constraints. The papers by Tribus (ref. 6) contain interesting arguments that support this idea.

The three cases presented for which the idea of maximum entropy was used are the following:

(1) Case 1: quiescent plasma

(2) Case 2: current-carrying plasma

(3) Case 3: viscous plasma

Case 1 is presented to illustrate the maximum entropy approach. The important point is that the analysis yields the conventional or accepted result for a simple case. Thus, the analysis gains credibility for application to more complex cases.

SYMBOLS

A determinant defined in eq. (45)

C_e	electron acoustic velocity (defined following eq. (40))
E_j	energy of j^{th} state as measured from continuum
E_0	energy of ground state as measured from continuum
e	electronic charge
F	density in phase space (six dimensional)
f	dimensionless distribution function
h	Planck's constant
J	current density (defined in eq. (32))
k	Boltzmann constant
M_e	electron diffusion Mach number (defined following eq. (40))
m_e, m_i, m_N	mass of electron, ion, and neutral atom ($m_i = m_N$)
n	number density
n_0	number density of nuclei ($n_0 = n_N + n_i$)
p	linear momentum density
S	entropy density of system (defined in eq. (1))
T	temperature
$\hat{\hat{U}}$	stress energy density (defined in eq. (41))
u	energy density defined in eq. (41))
x	fraction ionization
Z	partition function of electron states of neutrals
α_{kl}	function defined in eq. (47)
$\hat{\hat{\Gamma}}$	Lagrange multipliers conjugate to stress energy densities introduced in eq. (42)
ϵ_j	energy of j^{th} state
λ	Lagrange multiplier conjugate to energy density (reciprocal temperatures)
$\vec{\mu}$	Lagrange multiplier conjugate to momentum density
$\vec{\xi}$	velocity with respect to center of mass
Υ	dimensionless current (defined following eq. (36))
ψ	function defined in eq. (46)

Ω Lagrange multiplier conjugate to number density

ω degeneracy

Subscripts:

e electron

I internal energy

i ion

j j^{th} state

N neutral

Mathematical notation:

δ variation of

$d^3\xi$ shorthand for the three velocity coordinates $d\xi_1\, d\xi_2\, d\xi_3$

$\langle\rangle$ average value

$\rightarrow$ vector symbol

$\hat{\hat{}}$ dyadic symbol

$|\,|$ absolute value

ANALYSIS

For the purposes of this report, the definition of the entropy density is

$$S = -\omega\left(\frac{m}{h}\right)^3 \int f(\ln f - 1)\, d^3\xi \tag{1}$$

where ω is the degeneracy ($\omega = 2$ for electrons), m is the particle mass, h is Planck's constant, $d^3\xi$ is shorthand for the three velocity coordinates $d\xi_1$, $d\xi_2$, and $d\xi_3$, and f is the dimensionless distribution function.

The factor $\omega(m/h)^3$ arises from the correspondence principle (see Hill, ref. 7, p. 80) in passing to the classical limit (for free particles in a box) from the discrete to the continuous distribution function.

The relation between the distribution function f and the density F in phase space (six-dimensional) is given by

$$F = \omega\left(\frac{m}{h}\right)^3 f \tag{2}$$

The maximum density in phase space is $\omega(m/h)^3$ for the completely degenerate gas, for which case $f = 1$. The entropy, as given in equation (1), has a maximum at $f = 1$. Taking a variation of equation (1) gives

$$\delta S = -\omega\left(\frac{m}{h}\right)^3 \int \delta f \ln f \, d^3\xi = 0$$

or

$$f = 1$$

This entropy maximum is taken to be the state of greatest disorder, that is, the completely chaotic condition.

Grad (ref. 8) points out that a given system has many entropies depending on those properties that the observer wishes to study. Once the system and the relevant measurements are defined, any quarrel between two observers over the proper choice of entropy implies that they are interested in different phenomena.

The system under consideration in this study consists of a gas mixture of electrons, ions, and neutral atoms. (The neutrals are in various levels of excitation.) A series of measurements with respect to the system center of mass are assumed to yield a knowledge of various moments of the distribution function of each species. Moreover, it is assumed that an entropy function of the form given in equation (1) can be written for each species.

The entropy density of each species is maximized subject to the imposed constraints (i.e., the moments), the constraints are introduced by the method of Lagrange's undetermined multipliers. This introduction yields the most probable distribution for each species in terms of the undetermined multipliers. Since there is one undetermined multiplier for each constraint, the equations of constraint can be used in principle to determine the Lagrange multipliers. The distribution function of each species is then uniquely determined in terms of the assumed measurable information.

The thermodynamics of the study begins when the relations between the Lagrange multipliers of the various species are investigated. Either by hypothesizing or by reviewing the previous measurements, one can conclude, for example, that the system is charge neutral, that at each station the sum of the ion and the neutral densities is a constant, or that thermal equilibrium exists between free and bound electrons. This information, which is a subset of the measurements that led to the choice of constraints, is called the auxiliary information. The auxiliary equations are based on the usual plasma models, and the final results are limited to the accuracy of this model. The auxiliary equations, however, do not limit the validity of the maximum entropy viewpoint. They can be replaced by more accurate expressions with no alteration in the choice of constraints.

Analysis of Case 1: Quiescent Plasma

For the analysis of a quiescent plasma, a series of measurements with respect to the system center of mass is assumed. Through these measurements the following moments can be inferred:

Number densities:

$$n_e = 2\left(\frac{m_e}{h}\right)^3 \int f_e \, d^3\xi_e \tag{3a}$$

$$n_i = \omega_i\left(\frac{m_i}{h}\right)^3 \int f_i \, d^3\xi_i \tag{3b}$$

$$n_N = \sum_j n_j = \sum_j \omega_j\left(\frac{m_N}{h}\right)^3 \int f_j \, d^3\xi_N \tag{3c}$$

Kinetic energy densities:

$$u_e \equiv \frac{1}{2} m_e n_e \left\langle \xi_e^2 \right\rangle = 2\left(\frac{m_e}{h}\right)^3 \int f_e\left(\frac{1}{2} m_e \xi_e^2\right) d^3\xi_e \tag{4a}$$

$$u_i \equiv \frac{1}{2} m_i n_i \left\langle \xi_i^2 \right\rangle = \omega_i\left(\frac{m_i}{h}\right)^3 \int f_i\left(\frac{1}{2} m_i \xi_i^2\right) d^3\xi_i \tag{4b}$$

$$u_N \equiv \frac{1}{2} m_N n_N \left\langle \xi_N^2 \right\rangle = \sum_j \omega_j \left(\frac{m_N}{h}\right)^3 \int f_j \left(\frac{1}{2} m_N \xi_N^2 \, d^3 \xi_N\right) \tag{4c}$$

Internal energy density:

$$u_I \equiv \sum_j n_j \epsilon_j = \sum_j \omega_j \epsilon_j \left(\frac{m_N}{h}\right)^3 \int f_j \, d^3 \xi_N \tag{4d}$$

Equating the moments (eqs. (3) and (4)) to their measured values gives the set of constraint equations.

The entropy equations for the species are (from eq. (1))

$$S_e = -2\left(\frac{m_e}{h}\right)^3 \int f_e (\ln f_e - 1) \, d^3 \xi_e \tag{5a}$$

$$S_i = -\omega_i \left(\frac{m_i}{h}\right)^3 \int f_i (\ln f_i - 1) \, d^3 \xi_i \tag{5b}$$

$$S_N = -\sum_j \omega_j \left(\frac{m_N}{h}\right)^3 \int f_j (\ln f_j - 1) \, d^3 \xi_N \tag{5c}$$

and the variational equations are

$$\delta S_e - \Omega_e \delta n_e - \lambda_e \delta u_e = 0 \tag{6a}$$

$$\delta S_i - \Omega_i \delta n_i - \lambda_i \delta u_i = 0 \tag{6b}$$

$$\delta S_N - \Omega_N \delta n_N - \lambda_N \delta u_N - \lambda_I \delta u_I = 0 \tag{6c}$$

where Ω's and λ's are the Lagrange multipliers.

Substituting equations (3), (4), and (5) into equations (6) gives

$$-2\left(\frac{m_e}{h}\right)^3 \int \delta f_e\left(\ln f_e + \Omega_e + \lambda_e \frac{1}{2} m_e \xi_e^2\right) d^3\xi_e = 0 \tag{7a}$$

$$-\omega_i\left(\frac{m_i}{h}\right)^3 \int \delta f_i\left(\ln f_i + \Omega_i + \lambda_i \frac{1}{2} m_i \xi_i^2\right) d^3\xi_i = 0 \tag{7b}$$

$$-\sum_j \omega_j\left(\frac{m_N}{h}\right)^3 \int \delta f_j\left(\ln f_j + \Omega_N + \lambda_N \frac{1}{2} m_N \xi_N^2 + \lambda_I \epsilon_j\right) d^3\xi_N = 0 \tag{7c}$$

Because the variations are arbitrary,

$$f_e = e^{-\Omega_e - \lambda_e \, 1/2 \, m_e \xi_e^2} \tag{8a}$$

$$f_i = e^{-\Omega_i - \lambda_i \, 1/2 \, m_i \xi_i^2} \tag{8b}$$

$$f_j = e^{-\Omega_N - \lambda_N \, 1/2 \, m_N \xi_N^2 - \lambda_I \epsilon_j} \tag{8c}$$

Substituting equations (8) into equations (3) gives the number densities

$$n_e = 2\left(\frac{2\pi m_e}{\lambda_e h^2}\right)^{3/2} e^{-\Omega_e} \tag{9a}$$

$$n_i = \omega_i\left(\frac{2\pi m_i}{\lambda_i h^2}\right)^{3/2} e^{-\Omega_i} \tag{9b}$$

$$n_j = \omega_j \, e^{-\lambda_I \epsilon_j}\left(\frac{2\pi m_N}{\lambda_N h^2}\right)^{3/2} e^{-\Omega_N} \tag{9c}$$

The zero of energy for the bound states is taken as the zero of kinetic energy, so that the levels ϵ_j are negative ($\epsilon_j = -E_j$) in this frame of reference. Thus,

$$n_N = \sum_j n_j = \left(\frac{2\pi m_N}{\lambda_N h^2}\right)^{3/2} e^{-\Omega_N+\lambda_I E_0} Z(\lambda_I) \tag{10}$$

where

$$Z(\lambda_I) \equiv \sum_j \omega_j e^{-\lambda_I(E_0-E_j)} \tag{11}$$

is the partition function of the bound states, and for moderate temperatures its value is the ground state degeneracy ω_0.

Because

$$-\frac{\partial f_e}{\partial \lambda_e} = \frac{1}{2} m_e \xi_e^2 f_e \tag{12}$$

the energy densities can be written as

$$\frac{1}{2} m_e n_e \left\langle \xi_e^2 \right\rangle = -\frac{\partial n_e}{\partial \lambda_e} \tag{13}$$

From equations (4) and (9),

$$\frac{1}{\lambda_e} = \frac{1}{3} m_e \left\langle \xi_e^2 \right\rangle \equiv \frac{2}{3} \frac{u_e}{n_e} \tag{14a}$$

$$\frac{1}{\lambda_i} = \frac{1}{3} m_i \left\langle \xi_i^2 \right\rangle \equiv \frac{2}{3} \frac{u_i}{n_i} \tag{14b}$$

$$\frac{1}{\lambda_N} = \frac{1}{3} m_N \left\langle \xi_N^2 \right\rangle \equiv \frac{2}{3} \frac{u_N}{n_N} \tag{14c}$$

Equations (14) allow the identification of the λ's as reciprocal temperatures $1/kT$ (see ter Harr, ref. 9, p. 3). It is assumed that λ_I is also a reciprocal temperature. Combining equations (8) and (5) yields

$$S_e = (\Omega_e + 1)n_e + \lambda_e u_e = S_e(n_e, u_e) \tag{15a}$$

$$S_i = (\Omega_i + 1)n_i + \lambda_i u_i = S_i(n_i, u_i) \tag{15b}$$

$$S_N = (\Omega_N + 1)n_N + \lambda_N u_N + \lambda_I u_I = S(n_N, u_N, u_I) \tag{15c}$$

The Ω's and λ's are functions of the n's and u's because of the equations of constraint.

It is easily verified that

$$\left.\begin{array}{l} \left(\dfrac{\partial S_e}{\partial n_e}\right)_{u_e} = \Omega_e; \quad \left(\dfrac{\partial S_e}{\partial u_e}\right)_{n_e} = \lambda_e \\ \left(\dfrac{\partial S_i}{\partial n_i}\right)_{u_i} = \Omega_i; \quad \left(\dfrac{\partial S_i}{\partial u_i}\right)_{n_i} = \lambda_i \\ \left(\dfrac{\partial S_N}{\partial n_N}\right)_{u_N, u_I} = \Omega_N; \quad \left(\dfrac{\partial S_N}{\partial u_N}\right)_{n_N, u_I} = \lambda_N; \quad \left(\dfrac{\partial S_N}{\partial u_I}\right)_{n_N, u_N} = \lambda_I \end{array}\right\} \tag{16}$$

An arbitrary change in the system entropy is given by

$$dS = d(S_e + S_i + S_N) = \Omega_e\, dn_e + \Omega_i\, dn_i + \Omega_N\, dn_N + \lambda_e\, du_e + \lambda_i\, du_i + \lambda_N\, du_N + \lambda_I\, du_I \tag{17}$$

The auxiliary equations relating the densities of the various species are assumed to be a charge neutrality

$$n_e = n_i \tag{18a}$$

and a constant number density of nuclei

$$n_i + n_N = n_0 \tag{18b}$$

The relaxation time of the free and bound electron subsystem is much shorter than that between the heavy species and the electrons. Thus, in a plasma a metastable equilibrium may exist, which leads to the two-temperature model (see Kerrebrock, ref. 10). Because of these inherent relaxation times of the system, the following auxiliary equations relating the changes in the energy densities of the various species are reasonable:

$$du_e = -du_I \tag{19a}$$

$$du_i = -du_N \tag{19b}$$

Because the system entropy is also a maximum, dS must vanish, and equation (17), with substitutions from equations (18) and (19), becomes

$$(\Omega_e + \Omega_i - \Omega_N)\, dn_e + (\lambda_e - \lambda_I)\, du_e + (\lambda_i - \lambda_N)\, du_i = 0 \tag{20}$$

Since dn_e, du_e, and du_i are arbitrary nontrivial changes, it follows that

$$\Omega_e + \Omega_i = \Omega_N \tag{21a}$$

$$\lambda_e = \lambda_I \tag{21b}$$

$$\lambda_i = \lambda_N \tag{21c}$$

When equations (9a) and (b) and equation (10) are solved for the Ω's, equation (21a) becomes

$$\frac{n_0 x^2}{1 - x} = \frac{n_e n_i}{n_N} = \frac{2\omega_i}{Z(\lambda_e)} \left(\frac{2\pi m_e}{\lambda_e h^2} \right)^{3/2} e^{-\lambda_e E_0} \tag{22}$$

where equations (21b) and (21c) have been used in simplifying the relation, and the fraction ionization is $x = n_i/n_0$. This equation is the Saha relation as used in the two-temperature plasma model (ref. 10) in which the electron temperature is elevated over the gas temperature.

Analysis of Case 2: Current-Carrying Plasma

For the analysis of a current-carrying plasma, the measurements of the system are assumed to indicate that diffusion of the three species with respect to the center of mass is occurring. Thus, in addition to the number and energy densities (eqs. (3) and (4)), the following moments are present:

$$\vec{p}_e \equiv n_e m_e \left\langle \vec{\xi}_e \right\rangle = 2\left(\frac{m_e}{h}\right)^3 \int f_e m_e \vec{\xi}_e \, d\xi_e^3 \tag{23a}$$

$$\vec{p}_i \equiv n_i m_i \left\langle \vec{\xi}_i \right\rangle = \omega_i \left(\frac{m_i}{h}\right)^3 \int f_i m_i \vec{\xi}_i \, d\xi_i \tag{23b}$$

$$\vec{p}_N \equiv n_N m_N \left\langle \vec{\xi}_N \right\rangle = \sum_j \omega_j \left(\frac{m_N}{h}\right)^3 \int f_j m_N \vec{\xi}_N \, d\xi_N^3 \tag{23c}$$

The equations of constraint now include these moments set equal to their measured values. The variational equations are then

$$\delta S_e - \Omega_e \delta n_e - \vec{\mu}_e \cdot \delta \vec{p}_e - \lambda_e \delta u_e = 0 \tag{24a}$$

$$\delta S_i - \Omega_i \delta n_i - \vec{\mu}_i \cdot \delta \vec{p}_i - \lambda_i \delta u_i = 0 \tag{24b}$$

$$\delta S_N - \Omega_N \delta n_N - \vec{\mu}_N \cdot \delta \vec{p}_N - \lambda_N \delta u_N - \lambda_I \delta u_I = 0 \tag{24c}$$

which yield

$$f_e = e^{-\Omega_e - \vec{\mu}_e \cdot m_e \vec{\xi}_e - \lambda_e 1/2 \, m_e \xi_e^2} \tag{25a}$$

$$f_i = e^{-\Omega_i - \vec{\mu}_i \cdot m_i \vec{\xi}_i - \lambda_i 1/2 \, m_i \xi_i^2} \tag{25b}$$

$$f_j = e^{-\Omega_N - \vec{\mu}_N \cdot m_N \vec{\xi}_N - \lambda_N 1/2 \, m_N \xi_N^2 - \lambda_I \epsilon_j} \tag{25c}$$

These equations can be shown to represent Maxwellian distributions for each species in its own center-of-mass frame of reference. The corresponding number densities are

$$n_e = 2\left(\frac{2\pi m_e}{\lambda_e h^2}\right)^{3/2} e^{-\Omega_e+(m_e/2)(\mu_e^2/\lambda_e)} \tag{26a}$$

$$n_i = \omega_i\left(\frac{2\pi m_i}{\lambda_i h^2}\right)^{3/2} e^{-\Omega_i+(m_i/2)(\mu_i^2/\lambda_i)} \tag{26b}$$

$$n_j = \omega_j e^{-\lambda_i \epsilon_j}\left(\frac{2\pi m_N}{\lambda_N h^2}\right)^{3/2} e^{-\Omega_N+(m_N/2)(\mu_N^2/\lambda_N)} \tag{26c}$$

$$n_N = \sum_j n_j = \left(\frac{2\pi m_N}{\lambda_N h^2}\right)^{3/2} e^{-\Omega_N+(m_N/2)(\mu_N^2/\lambda_N)+\lambda_I E_0} Z(\lambda_I) \tag{26d}$$

where $Z(\lambda_I)$ is the partition function defined in equation (11).

Equations (23) yield

$$\left\langle \vec{\xi}_e \right\rangle = -\frac{\vec{\mu}_e}{\lambda_e} \tag{27a}$$

$$\left\langle \vec{\xi}_i \right\rangle = -\frac{\vec{\mu}_i}{\lambda_i} \tag{27b}$$

$$\left\langle \vec{\xi}_N \right\rangle = -\frac{\vec{\mu}_N}{\lambda_N} \tag{27c}$$

and the energy density equations (see eq. (4)) yield

$$\frac{1}{\lambda_e} = \frac{m_e}{3}\left(\left\langle \xi_e^2 \right\rangle - \left\langle \vec{\xi}_e \right\rangle^2\right) \tag{28a}$$

$$\frac{1}{\lambda_i} = \frac{m_i}{3}\left(\left\langle \xi_e^2 \right\rangle - \left\langle \vec{\xi}_e \right\rangle^2\right) \tag{28b}$$

$$\frac{1}{\lambda_N} = \frac{m_N}{3}\left(\left\langle \xi_N^2 \right\rangle - \left\langle \vec{\xi}_N \right\rangle^2\right) \tag{28c}$$

Thus the λ's are again measures of the mean random energy and can be identified as reciprocal temperatures (1/kT) (see e.g. ter Haar, ref. 9, p. 17). Again it is assumed that λ_I is also a reciprocal temperature of the bound states.

The equations for the entropy of each species for the current-carrying case become

$$S_e = (\Omega_e + 1)n_e + \vec{\mu}_e \cdot \vec{p}_e + \lambda_e u_e \tag{29a}$$

$$S_i = (\Omega_i + 1)n_i + \vec{\mu}_i \cdot \vec{p}_i + \lambda_i u_i \tag{29b}$$

$$S_N = (\Omega_N + 1)n_N + \vec{\mu}_N \cdot \vec{p}_N + \lambda_N u_N + \lambda_I u_I \tag{29c}$$

where the Ω's, λ's, and μ's are functions of the n's, u's, and p's through the equations of constraint. As in case 1, it can be shown that

$$\left(\frac{\partial S_e}{\partial n_e}\right) = \Omega_e$$

$$\left(\frac{\partial S_e}{\partial u_e}\right) = \lambda_e$$

$$\mathrm{grad}_{p_e} S_e = \vec{\mu}_e$$

and similar identities for the other species. An arbitrary change in the system entropy

$$S = S_e + S_i + S_N$$

is then

$$dS = \Omega_e\, dn_e + \Omega_i\, dn_i + \Omega_N\, dn_N + \vec{\mu}_e \cdot d\vec{p}_i + \vec{\mu}_i \cdot d\vec{p}_i + \vec{\mu}_N \cdot d\vec{p}_N$$
$$+ \lambda_e\, du_e + \lambda_i\, du_i + \lambda_N\, du_N + \lambda_I\, du_I \tag{30}$$

The auxiliary equations for case 2 are assumed to be the same as those for case 1 (see eqs. (18) and (19)). In addition, from the definition of the center-of-mass velocity, it follows that

$$\vec{p}_e + \vec{p}_i + \vec{p}_N = 0 \tag{31}$$

Because of the diffusion among species, charges will be transported and cause a current density

$$\vec{J} = e\left(\frac{\vec{p}_i}{m_i} - \frac{\vec{p}_e}{m_e}\right) \tag{32}$$

Note that the electric field causing the diffusion of charges has been tacitly assumed. Any effect of the electric field other than the diffusion of charges is disregarded.

With the auxiliary equations substituted, equation (30) (for a constant current density) becomes

$$(\Omega_e + \Omega_i - \Omega_N)\, dn_e + \frac{(m_e\vec{\mu}_e + m_i\vec{\mu}_i - m_N\vec{\mu}_N)}{m_e} \cdot d\vec{p}_e + (\lambda_e - \lambda_I)\, du_e + (\lambda_i - \lambda_N)\, du_i = 0 \tag{33}$$

from which it is concluded that

$$\Omega_e + \Omega_i = \Omega_N \tag{34a}$$

$$m_e\vec{\mu}_e + m_i\vec{\mu}_i = m_N\vec{\mu}_N \tag{34b}$$

$$\lambda_e = \lambda_I \tag{34c}$$

$$\lambda_i = \lambda_N \tag{34d}$$

Equations (34) and (26) give the equivalent of the Saha equation for the flowing system as

$$\frac{n_e n_i}{n_N} = \frac{x^2 n_0}{(1 - x)} = \frac{2\omega_i e^{-\lambda_e E_0}}{Z(\lambda_e)} \left(\frac{2\pi m_e}{\lambda_e h^2}\right)^{3/2} e^{\Upsilon^2/x^2} \tag{35}$$

where

$$\frac{\Upsilon^2}{x^2} = \frac{m_e \mu_e^2}{2\lambda_e} + \frac{m_i \mu_i^2}{2\lambda_i} - \frac{m_N \mu_N^2}{2\lambda_N} \tag{36}$$

The fraction ionization is

$$x = \frac{n_i}{n_0}$$

and the dimensionless current is

$$\Upsilon = \left(\frac{m_e \lambda_e}{2}\right)^{1/2} \frac{|\vec{J}|}{en_0}$$

Equations (31), (32), and (34b) can be used to write equation (36) as

$$\frac{\Upsilon^2}{x^2} = \frac{m_e \lambda_e J^2}{2e^2 n_e^2} \left\{ \frac{1 + (1 - 2x) \dfrac{m_e}{m_N} \dfrac{\lambda_e}{\lambda_N} + 2x \dfrac{m_e}{m_N}}{\left[1 + (1 - x) \dfrac{m_e}{m_N} \dfrac{\lambda_e}{\lambda_N} + x \dfrac{m_e}{m_N}\right]^2} \right\} \tag{37}$$

Because of the mass difference between the electrons and the heavy species, equation (37) reduces to

$$\frac{\Upsilon^2}{x^2} \cong \frac{m_e \lambda_e J^2}{2e^2 n_e^2} \tag{38}$$

Case 2 can be compared with case 1 if equation (35) is combined with the Saha equation given in equation (22) (with the subscript s denoting Saha added). Thus,

$$\frac{x^2}{x_s^2}\,\frac{(1 - x_s)}{(1 - x)} = e^{\Upsilon^2/x^2} \tag{39}$$

Equation (38) can be written in terms of the electron diffusion velocity and Mach number as

$$\frac{\Upsilon^2}{x^2} \cong \frac{m_e\lambda_e}{2}\left\langle\vec{\xi}_e\right\rangle^2 = \frac{5}{6}\,M_e^2 \tag{40}$$

where

$$M_e \equiv \frac{\left|\left\langle\vec{\xi}_e\right\rangle\right|}{C_e}$$

and the electron acoustic velocity is

$$C_e = \left(\frac{5}{3m_e\lambda_e}\right)^{1/2}$$

Analysis of Case 3: Viscous Plasma

A viscous fluid is characterized as one having a nondiagonal pressure tensor. For the case under consideration, the relevant constraints are the number densities (eqs. (3)), the momentum densities (eqs. (23)), the internal energy density of the bound states (eq. (4d)), and the stress energy densities defined in the following equations:

$$\hat{\hat{U}}_e \equiv n_e m_e\left\langle\vec{\xi}_e\vec{\xi}_e\right\rangle = 2\left(\frac{m_e}{h}\right)^3\int f_e m_e\vec{\xi}_e\vec{\xi}_e\,d\xi_e^3 \tag{41a}$$

$$\hat{\hat{U}}_i \equiv n_i m_i\left\langle\vec{\xi}_i\vec{\xi}_i\right\rangle = \omega_i\left(\frac{m_i}{h}\right)^3\int f_i m_i\vec{\xi}_i\vec{\xi}_i\,d\xi_i^3 \tag{41b}$$

$$\hat{\hat{U}}_N \equiv n_N m_N \left\langle \vec{\xi}_N \vec{\xi}_N \right\rangle = \sum_j \omega_j \left(\frac{m_N}{h}\right)^3 \int f_j m_N \vec{\xi}_N \vec{\xi}_N \, d\xi_N^3 \tag{41c}$$

The variational equations are

$$\delta S_e - \Omega_e \delta n_e - \vec{\mu}_e \cdot \delta \vec{p}_e - \hat{\hat{\Gamma}}_e \cdot\cdot \delta \hat{\hat{U}}_e = 0 \tag{42a}$$

$$\delta S_i - \Omega_i \delta n_i - \vec{\mu}_i \cdot \delta \vec{p}_i - \hat{\hat{\Gamma}}_i \cdot\cdot \delta \hat{\hat{U}}_i = 0 \tag{42b}$$

$$\delta S_N - \Omega_N \delta n_N - \vec{\mu}_N \cdot \delta \vec{p}_N - \hat{\hat{\Gamma}}_N \cdot\cdot \delta \hat{\hat{U}}_N - \lambda_I \delta u_I = 0 \tag{42c}$$

The distribution functions are

$$f_e = e^{-\Omega_e - \vec{\mu}_e \cdot m_e \vec{\xi}_e - m_e \vec{\xi}_e \cdot \hat{\hat{\Gamma}}_e \cdot \vec{\xi}_e} \tag{43a}$$

$$f_i = e^{-\Omega_i - \vec{\mu}_i \cdot m_i \vec{\xi}_i - m_i \vec{\xi}_i \cdot \hat{\hat{\Gamma}}_i \cdot \vec{\xi}_i} \tag{43b}$$

$$f_j = e^{-\Omega_N - \vec{\mu}_N \cdot m_N \vec{\xi}_N - m_N \vec{\xi}_N \cdot \hat{\hat{\Gamma}}_N \cdot \vec{\xi}_N - \lambda_I \epsilon_j} \tag{43c}$$

The corresponding densities are

$$n_e = 2\left(\frac{m_e \pi}{h^2}\right)^{3/2} A_e^{-1/2} e^{\psi_e - \Omega_e} \tag{44a}$$

$$n_i = \omega_i \left(\frac{m_i \pi}{h^2}\right)^{3/2} A_i^{-1/2} e^{\psi_i - \Omega_i} \tag{44b}$$

$$n_j = \omega_j e^{-\lambda_I \epsilon_j} \left(\frac{m_N \pi}{h^2}\right)^{3/2} A_N^{-1/2} e^{\psi_N - \Omega_N} \tag{44c}$$

$$n_N = \sum_j n_j = e^{\lambda_I E_0} Z(\lambda_I)\left(\frac{m_N \pi}{h^2}\right)^{3/2} A_N^{-1/2} e^{\psi_N - \Omega_N} \tag{44d}$$

where

$$A = \det\begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\ \Gamma_{12} & \Gamma_{22} & \Gamma_{23} \\ \Gamma_{13} & \Gamma_{23} & \Gamma_{33} \end{bmatrix} \tag{45}$$

and

$$\psi = \frac{m}{4A}\left(\mu_1^2\alpha_{11} + \mu_2^2\alpha_{22} + \mu_3^2\alpha_{33} + \mu_1\mu_2\alpha_{12} + \mu_1\mu_3\alpha_{13} + \mu_2\mu_3\alpha_{23}\right) \tag{46}$$

where

$$\alpha_{kl} = \frac{\partial A}{\partial \Gamma_{kl}} \tag{47}$$

Arguments similar to those of cases 1 and 2 lead to the following result:

$$\frac{n_e n_i}{n_N} = \frac{2\omega_i e^{-\lambda_I E_0}}{Z(\lambda_I)}\left(\frac{A_N}{A_e A_i}\right)^{1/2}\left(\frac{m_e \pi}{h^2}\right)^{3/2} e^{\psi_e + \psi_i - \psi_N} \tag{48}$$

Equation (48) is written in terms of all the Lagrange multipliers; only diffusional equilibrium is assumed (eq. (34a)). Simplifying relations that have not been investigated may exist.

RESULTS AND DISCUSSION

The principle of maximum entropy (disorder) has been applied to three cases of plasma behavior: quiescent, current carrying, and viscous. The following Saha-type equations were found:

$$\frac{n_e n_i}{n_N} = n_0 \frac{x^2}{1 - x} = \frac{2\omega_i e^{-\lambda_e E_0}}{Z(\lambda_e)} \left(\frac{2\pi m_e}{\lambda_e h^2}\right)^{3/2} \tag{22}$$

$$\frac{n_e n_i}{n_N} = \frac{2\omega_i e^{-\lambda_e E_0}}{Z(\lambda_e)} \left(\frac{2\pi m_e}{\lambda_e h^2}\right)^{3/2} e^{\Upsilon^2/x^2} \tag{35}$$

$$\frac{n_e n_i}{n_N} = \frac{2\omega_i e^{-\lambda_I E_0}}{Z(\lambda_I)} \left(\frac{A_N}{A_e A_i}\right)^{1/2} \left(\frac{m_e \pi}{h^2}\right)^{3/2} e^{\psi_e + \psi_i - \psi_N} \tag{48}$$

The results of case 1 are, of course, well known. This case was presented as a simple illustration of the procedure used in the later sections of this report.

Equation (39) from case 2 is plotted in figure 1. The parameter Υ is a dimensionless current, and x is the fraction ionization n_i/n_0 of the gas. The intercepts are the normal Saha conditions (case 1) for the same temperature and density of nuclei n_0.

The line of the electron Mach number, $M_e = 1$, drawn in figure 1 indicates that an effective increase in fraction ionization occurs only in situations in which the directed energy of the electrons is a substantial fraction of the total energy. However, in such highly directed streams, other factors such as instabilities may contribute to modify the results.

With the acknowledgment of these limitations and qualifications, it is interesting to examine the application of these results to cesium and argon plasmas. Figures 2(a) and (b) are plots of the fraction ionization as a function of the actual current density for various initial fraction ionizations for cesium and argon, respectively. A nuclei density of 10^{20} per cubic meter is assumed in both these figures. In figure 3, the fraction ionization as a function of nuclei densities is compared for cesium and argon. The Saha tabulations of Drawin (ref. 11) were used to make these plots.

The results of case 3 (viscous plasma) are cursory, and they are presented only to point out another possible application of the maximum entropy method. Further reduction of this case is beyond the scope of this report.

LIMITATIONS AND CONCLUDING REMARKS

The primary limitation of the method presented in this report is its restriction to mixtures of ideal gases. Extension of the method to nonideal gases would entail an entropy function that would take into account correlations among particles.

The second limitation is the choice of auxiliary equations that enable one to find relations among the Lagrange multipliers of each species. If there are no interactions between species, then there will be no relations (except accidental ones) between the Lagrange multipliers. The distribution functions in this case would be independent of each other. In a sense, the auxiliary equations are the results of interactions, and a rigorous treatment would demand solving for the kinetic rate processes. The choice of auxiliary equations used in this report are therefore subject to review in any real plasma problem.

The justification for the assumption that the free and bound electrons exchange energy to the exclusion of electron-heavy particle interactions (and radiation losses) is based on previous studies. Kerrebrock (ref. 10) discusses the electron heating phenomena and the two-temperature plasma model. The validity limits of the Saha equation for radiationless plasmas are discussed by Griem (ref. 12) and also in a previous publication by the author of this report (ref. 13). These studies are based on the comparison of lifetimes of atomic states and are thus beyond the realm of thermodynamics. It is sufficient to know that such regimes exist.

The study of the current-carrying plasma in case 2 is based on the same assumptions used for case 1. The effect of the current flow is an enhancement of Saha's equation (see eq. (39) and fig. 1). The current densities at which an appreciable effect occurs are much higher than those currents considered by other authors. For this reason, a comparison of the thermodynamic result to the kinetic approach of Dugan, et al. (ref. 14) is not possible. Likewise, the measurements made by Solbes (ref. 15) are not of a sufficiently high current density to prove or disprove the effect. With respect to figure 1, it is likely that one would have to look for the effect in shock waves, in high density arcs, or in some other device in which the directed energy of the electrons is great.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, March 6, 1968,
129-02-01-07-22.

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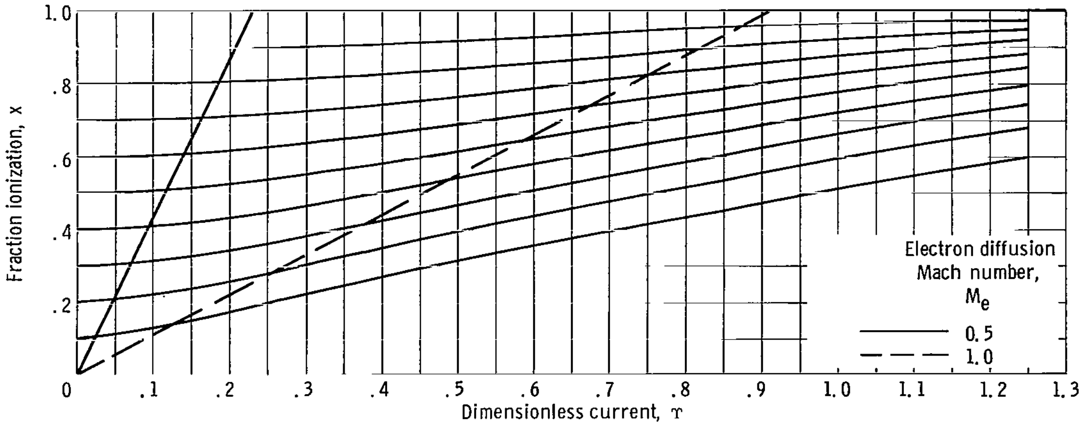
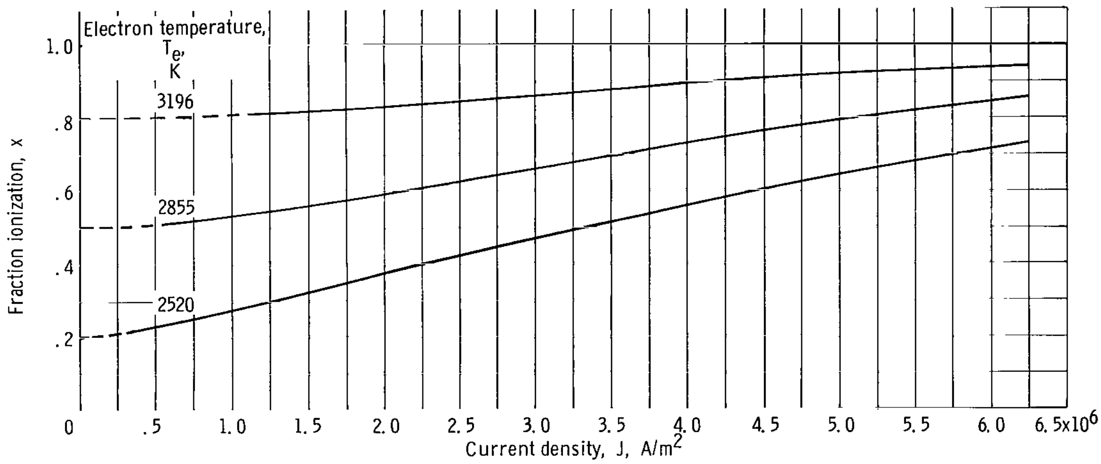
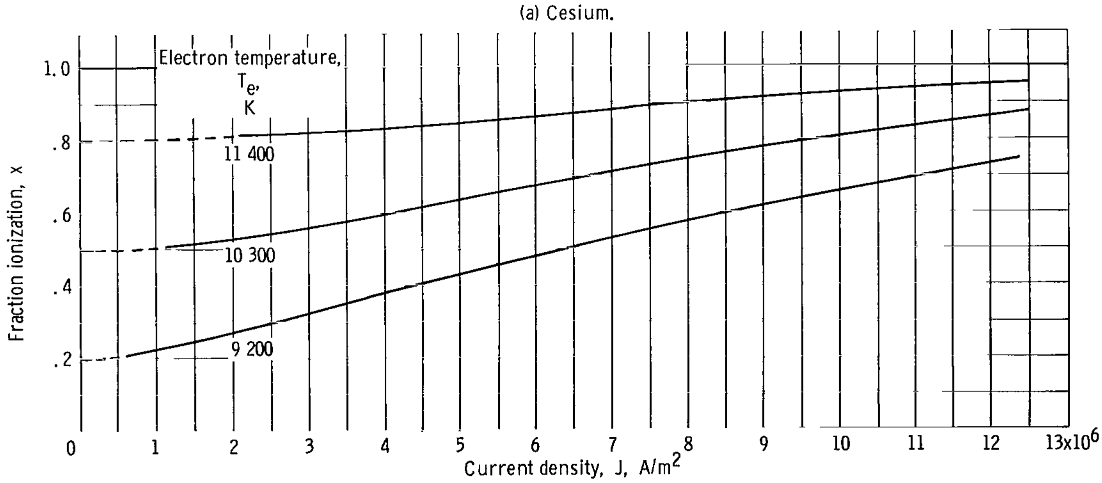


Figure 1. - Fraction ionization as function of normalized current density for various initial fraction ionizations.



(a) Cesium.



(b) Argon.

Figure 2. - Fraction ionization as function of current density for various initial fraction ionizations with nuclei density of 10^{20} per cubic meter.

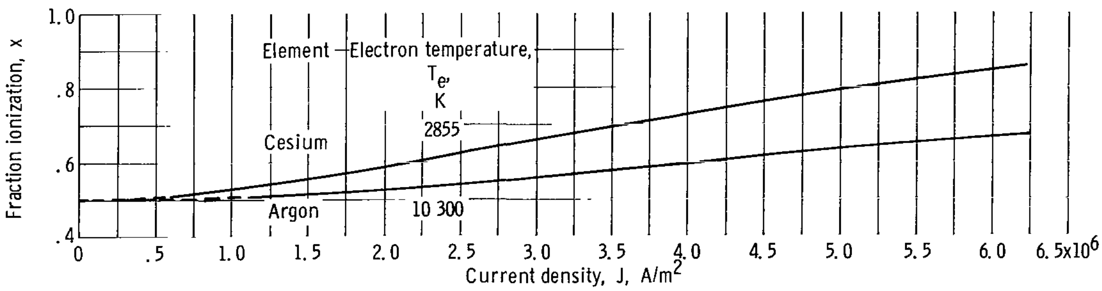


Figure 3. - Comparison of cesium and argon fraction ionization as function of current density with nuclei densities of 10^{20} per cubic meter at initial fraction ionization of 0.5.

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